

Higher Dimensional Multiparameter Unitary and Nonunitary Braid Matrices: Even Dimensions

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Abstract

A class of $(2n)^2 \times (2n)^2$ multiparameter braid matrices are presented for all n ($n \geq 1$). Apart from the spectral parameter θ , they depend on $2n^2$ free parameters $m_{ij}^{(\pm)}$, $i, j = 1, \dots, n$. For real parameters the matrices $R(\theta)$ are nonunitary. For purely imaginary parameters they became unitary. Thus a unification is achieved with odd dimensional multiparameter solutions presented before.

1 Introduction

Higher dimensional unitary braid matrices have been studied in two recent papers [1, 2]. Their simultaneous relevance to topological and quantum entanglements (as discussed, for example, in Ref. [3]) was a major motivation. In Ref. [2] quite different classes were presented for odd and even dimensional matrices. There, the even dimensional $(2n)^2 \times (2n)^2$ braid matrices have no free parameter (apart from the spectral parameter θ after Baxterization) where as the $(2n+1)^2 \times (2n+1)^2$ matrices have $2n(n+2)$ free parameters $(m_{ij}^{(\pm)})$. Here we unify the two cases by presenting multiparameter solutions

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for even dimensions. We obtain first the general case for this class and then show how to implement unitarity.

2 Constructions (Even dimensions)

The braid equation is, in standard notations, in presence of a spectral parameter θ ,

$$\widehat{R}_{12}(\theta) \widehat{R}_{23}(\theta + \theta') \widehat{R}_{12}(\theta') = \widehat{R}_{23}(\theta') \widehat{R}_{12}(\theta + \theta') \widehat{R}_{23}(\theta), \quad (2.1)$$

where $\widehat{R}_{12} = \widehat{R} \otimes I$ and $\widehat{R}_{23} = I \otimes \widehat{R}$. We present below a simple class of multiparameter solutions for $(2n)^2 \times (2n)^2$ ($n \geq 1$) braid matrices $\widehat{R}(\theta)$. They are analogous to the odd dimensional solutions presented before [4]. Unitarity constraints can be implemented as in sec. 5 of Ref. [2]. Thus, for this class, one obtains a unified approach for multiparameter odd and even dimensional solutions.

Define the projectors

$$P_{ij}^{(\epsilon)} = \frac{1}{2} \{ (ii) \otimes (jj) + (\bar{i}\bar{i}) \otimes (\bar{j}\bar{j}) + \epsilon [(\bar{i}\bar{i}) \otimes (j\bar{j}) + (\bar{i}i) \otimes (\bar{j}j)] \}, \quad (2.2)$$

where $i, j \in \{1, \dots, n\}$, $\bar{i} = 2n + 1 - i$, $\bar{j} = 2n + 1 - j$ and $\epsilon = \pm$. They provide a complete basis satisfying

$$P_{ij}^{(\epsilon)} P_{kl}^{(\epsilon')} = \delta_{ik} \delta_{jl} \delta_{\epsilon\epsilon'} P_{ij}^{(\epsilon)}, \quad \sum_{\epsilon=\pm} \sum_{i,j=1}^n P_{ij}^{(\epsilon)} = I_{(2n)^2 \times (2n)^2}. \quad (2.3)$$

Anticipating the basic constraints essential for odd dimension [4] we directly postulate the form

$$\widehat{R}(\theta) = \sum_{\epsilon=\pm} \sum_{i,j=1}^n e^{m_{ij}^{(\epsilon)} \theta} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right). \quad (2.4)$$

The proof that it satisfies the braid equation (2.1) proceeds in close analogy to the equations from (A9) to (A17) of Ref. [4]. Here we have directly implemented the constraint

$$m_{ij}^{(\epsilon)} = m_{i\bar{j}}^{(\epsilon)}. \quad (2.5)$$

It is instructive to study explicitly the simplest cases.

Case 1: $N = 2$ ($n = 1$) (Here $i = 1$, $\bar{i} = 2$ and similarly for j).

$$\begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}, \quad (2.6)$$

with

$$a_{\pm} = \frac{1}{2} \left(e^{m_{11}^{(+)} \theta} \pm e^{m_{11}^{(-)} \theta} \right). \quad (2.7)$$

Case 2: $N = 4$ ($n = 2$) (Here $i = 1, 2, \bar{i} = 3, 4$). In terms of 4×4 blocks (D_{ij}, A_{ij} on the diag. and anti-diag. respectively)

$$\begin{pmatrix} D_{11} & 0 & 0 & A_{1\bar{1}} \\ 0 & D_{22} & A_{2\bar{2}} & 0 \\ 0 & A_{\bar{2}2} & D_{\bar{2}\bar{2}} & 0 \\ A_{\bar{1}1} & 0 & 0 & D_{\bar{1}\bar{1}} \end{pmatrix}, \quad (2.8)$$

with

$$\begin{aligned} D_{11} = D_{\bar{1}\bar{1}} &= \begin{pmatrix} a_+ & 0 & 0 & 0 \\ 0 & b_+ & 0 & 0 \\ 0 & 0 & b_+ & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix}, & D_{22} = D_{\bar{2}\bar{2}} &= \begin{pmatrix} c_+ & 0 & 0 & 0 \\ 0 & d_+ & 0 & 0 \\ 0 & 0 & d_+ & 0 \\ 0 & 0 & 0 & c_+ \end{pmatrix}, \\ A_{1\bar{1}} = A_{\bar{1}1} &= \begin{pmatrix} 0 & 0 & 0 & a_- \\ 0 & 0 & b_- & 0 \\ 0 & b_- & 0 & 0 \\ a_- & 0 & 0 & 0 \end{pmatrix}, & A_{2\bar{2}} = A_{\bar{2}2} &= \begin{pmatrix} 0 & 0 & 0 & c_- \\ 0 & 0 & d_- & 0 \\ 0 & d_- & 0 & 0 \\ c_- & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} a_{\pm} &= \frac{1}{2} \left(e^{m_{11}^{(+)}\theta} \pm e^{m_{11}^{(-)}\theta} \right), & b_{\pm} &= \frac{1}{2} \left(e^{m_{12}^{(+)}\theta} \pm e^{m_{12}^{(-)}\theta} \right), \\ c_{\pm} &= \frac{1}{2} \left(e^{m_{21}^{(+)}\theta} \pm e^{m_{21}^{(-)}\theta} \right), & d_{\pm} &= \frac{1}{2} \left(e^{m_{22}^{(+)}\theta} \pm e^{m_{22}^{(-)}\theta} \right) \end{aligned} \quad (2.10)$$

We have verified, using a program, the braid equation (2.1) by inserting (2.4) for $N = 2, 4, 6, 8$. These provide direct checks for the argument indicated below (2.4). As compared to $\frac{1}{2}(N+3)(N-1)$ free parameters for odd [4], here for even N we obtain $\frac{1}{2}N^2$ free parameters $m_{ij}^{(\pm)}$.

Let us just note that the odd dimensional solutions of Ref. [4] and the even dimensional solutions presented here can be regrouped in a single expression given by

$$\hat{R}(\theta) = \frac{1}{2} \sum_{\epsilon=\pm} \sum_{i,j=1}^N e^{m_{ij}^{(\epsilon)}\theta} [(ii) \otimes (jj) + \epsilon(i\bar{i}) \otimes (j\bar{j})], \quad (2.11)$$

where

$$\begin{aligned} m_{ij}^{(\epsilon)} &= m_{\bar{i}\bar{j}}^{(\epsilon)} = m_{i\bar{j}}^{(\epsilon)} = m_{\bar{i}j}^{(\epsilon)}, & i, j &= 1, \dots, N \text{ and } \epsilon = \pm 1, \\ \overline{n+1} &= n+1 \text{ and } m_{n+1, n+1}^{(\epsilon)} = 0 (\forall \epsilon) & \text{If } N \text{ is odd, i.e. } N = 2n+1 \end{aligned} \quad (2.12)$$

3 Unitarity

For all parameter real, $\hat{R}(\theta)$ is real but not unitary. Exactly as for N odd, making each exponent purely imaginary, namely $\exp(m_{ij}^{(\pm)}\theta) \longrightarrow \exp(\mathbf{i}m_{ij}^{(\pm)}\theta)$, where on the right

$m_{ij}^{(\pm)}\theta$ is now real with a coefficient \mathbf{i} ($\mathbf{i}^2 = -1$), one obtains unitarity. Now, due to the symmetry of the projectors

$$\hat{R}(\theta)^+ = \hat{R}(-\theta) = \hat{R}(\theta)^{-1}, \quad \hat{R}(\theta)^+ \hat{R}(\theta) = I_{(2n)^2 \times (2n)^2}. \quad (3.1)$$

In general, one can demonstrate that our multiparameter odd and even dimensional solutions one has a simple factorization

$$\hat{R}(\theta_1 \pm \theta_2) = \hat{R}(\theta_1) \hat{R}(\pm\theta_2) = \hat{R}(\theta_1) \hat{R}(\theta_2)^{\pm 1}. \quad (3.2)$$

This evidently, holds for real or imaginary parameters, i.e. for nonunitary and unitary solutions. Correspondingly, the $R\mathbf{T}\mathbf{T}$ relations can be expressed as follows:

$$\left(\hat{R}(\theta) (\mathbf{T}(\theta) \otimes I) \right) \left((I \otimes \mathbf{T}(\theta')) \hat{R}(\theta') \right) = \left(\hat{R}(\theta') (\mathbf{T}(\theta') \otimes I) \right) \left((I \otimes \mathbf{T}(\theta)) \hat{R}(\theta) \right). \quad (3.3)$$

For comparison one may note that the real unitary braid matrix for all N ($= 2n$) presented in [2] can be written as

$$\hat{R}(z) = \left(\frac{1 - \mathbf{i}z}{1 + \mathbf{i}z} \right)^{1/2} P_+ + \left(\frac{1 + \mathbf{i}z}{1 - \mathbf{i}z} \right)^{1/2} P_-, \quad (3.4)$$

where

$$z = \tanh(\theta), \quad \left(\frac{1 \mp \mathbf{i}z}{1 \pm \mathbf{i}z} \right)^{1/2} \equiv e^{\pm \mathbf{i}\varphi}, \quad (3.5)$$

say, giving phases for the coefficients and

$$P_{\pm} = \frac{1}{2} (I \otimes I \pm \mathbf{i}K \otimes J). \quad (3.6)$$

(K, J being given by (2.2) of Ref. [2]). P_{\pm} can be expressed as sums of the projectors of the type

$$Q_{ij}^{(\epsilon)} = \frac{1}{2} \left\{ (ii) \otimes (jj) + (\bar{i}\bar{i}) \otimes (\bar{j}\bar{j}) + \epsilon \mathbf{i}(-1)^{\bar{j}} [(\bar{i}\bar{i}) \otimes (j\bar{j}) - (\bar{i}\bar{i}) \otimes (\bar{j}j)] \right\} \quad (3.7)$$

defining analogously $Q_{i\bar{j}}^{(\epsilon)}$ (with $j \longrightarrow \bar{j}$, $\bar{j} \longrightarrow j$ in $Q_{ij}^{(\epsilon)}$). The imaginary factor \mathbf{i} in $Q_{ij}^{(\epsilon)}$, $Q_{i\bar{j}}^{(\epsilon)}$ and the phases cancel giving a real $\hat{R}(z)$,

$$\hat{R}(z) = I \otimes I + zK \otimes J. \quad (3.8)$$

Due to the summing up of $Q_{ij}^{(\epsilon)}$ into P_{\pm} the effective number of projectors does not increase with N , nor the number of parameters. Here we have presented a class of solutions where the number of free parameters increase as N^2 . For this case, one can prove that

$$\hat{R}(z_1) \hat{R}(z_2) = \hat{R}(z_3), \quad (3.9)$$

where

$$z_3 = \frac{z_1 + z_2}{1 - z_1 z_2} \neq \tanh(\theta_1 + \theta_2), \quad (z_1 z_2 \neq 1). \quad (3.10)$$

4 θ -Expansion

In section 5 of Ref. [4] exponentiation and θ -expansion of $\hat{R}(\theta)$ was presented for odd dimension. We present below a brief analogous treatment for even dimensions. One have

$$e^{m_{ij}^{(\epsilon)}\theta} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right) = \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(m_{ij}^{(\epsilon)} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right) \right)^k \theta^k. \quad (4.1)$$

Defining

$$X = \sum_{\epsilon=\pm} \sum_{i,j=1}^n m_{ij}^{(\epsilon)} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right) \implies X^n = \sum_{\epsilon=\pm} \sum_{i,j=1}^n \left(m_{ij}^{(\epsilon)} \right)^n \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right) \quad (4.2)$$

due to the orthogonality of the projectors $\left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right)$ for different sets of indices. Now from (2.4), due to the completeness (2.3),

$$\hat{R}(\theta) = \sum_{\epsilon=\pm} \sum_{i,j=1}^n e^{m_{ij}^{(\epsilon)}\theta} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right) = I + \sum_{k=1}^{\infty} \frac{1}{k!} X^k \theta^k = e^{\theta X}. \quad (4.3)$$

Hence the braid equation (2.1) reduces to

$$e^{\theta X_{12}} e^{(\theta+\theta')X_{23}} e^{\theta' X_{12}} = e^{\theta' X_{23}} e^{(\theta+\theta')X_{12}} e^{\theta X_{23}}, \quad (4.4)$$

where $X_{12} = X \otimes I$ and $X_{23} = I \otimes X$. Expanding both sides and comparing coefficients of $\theta^a (\theta + \theta')^b \theta'^c$ one obtains a sequence of relations involving X_{12} , X_{23} . Some have been pointed out in section 5 of Ref. [4]. There would be parallel features here.

After implementing unitarity as in section 3 one can define (with \mathbf{i} as above (3.1))

$$X = \mathbf{i} \sum_{i,j,\epsilon} m_{ij}^{(\epsilon)} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)} \right). \quad (4.5)$$

One then proceeds as above.

We have started with multiparameter case. For

$$\hat{R}(z) = I \otimes I + zK \otimes J = \left(\frac{1 - \mathbf{i}z}{1 + \mathbf{i}z} \right)^{1/2} P_+ + \left(\frac{1 + \mathbf{i}z}{1 - \mathbf{i}z} \right)^{1/2} P_- \equiv e^{\mathbf{i}\varphi} P_+ + e^{-\mathbf{i}\varphi} P_-. \quad (4.6)$$

(see the discussion following (3.3))

$$X = \mathbf{i}(P_+ - P_-) = -K \otimes J \quad (4.7)$$

and

$$\hat{R}(z) = \hat{R}(\varphi) = e^{\varphi X}, \quad (4.8)$$

where $e^{\mathbf{i}\varphi} = \left(\frac{1 - \mathbf{i} \tanh \theta}{1 + \mathbf{i} \tanh \theta} \right)^{1/2}$. By inserting this X in (4.4), one can develop in φ as explained before.

5 Discussion

For the multiparameter solutions presented in sections 2 and 3 one can study \hat{R} TT relations, transfer matrices, Hamiltonians and factorizable S -matrices in a closely analogous fashion to that for odd dimensions [5]. They are beyond the scope of this paper, limited essentially to construction of multiparameter $(2n)^2 \times (2n)^2$ braid matrices (nonunitary and unitary).

Beyond the unification presented there is a basic difference between odd and even dimensional cases. For the $(2n+1)^2 \times (2n+1)^2$ braid matrices with a basis of our "nested sequence" of projectors our multiparameter solutions are the most general ones. The presence of the central element 1 in $\hat{R}(\theta)$ imposes the simple exponential solutions for the coefficients of all other projectors. This has been emphasized in appendix A of Ref. [4]. ("Solving the braid equation"). But for even dimension this not the case. The class of solutions presented here is only one possibility. Already for the 4×4 case the intensively studied 6- and 8-vertex solutions can be canonically expressed on our basis (sections 6 and 7 of Ref. [6]). The multidimensional generalization of the 6-vertex matrix presented in Ref. [7] (citing previous sources) remains restricted to a single parameter γ . Are there authentic multiparameter generalizations of 6- and 8-vertex solutions to $(2n)^2 \times (2n)^2$ matrices for $n > 1$? We intend to explore such possibilities elsewhere.

We point out moreover that a pure imaginary spectral parameter ($\theta \rightarrow i\theta$) renders the 6-vertex and 8-vertex braid matrices unitary. This particularly evident from the respective canonical forms ((6.5) for 6-vertex and (7.2) with (7.6), (7.7) for 8-vertex of Ref. [6]) where the normalization factors are also suitably adapted. The coefficient of each real symmetric projector is evidently inverted under conjugation ($i\theta \rightarrow -i\theta$). Hence

$$\hat{R}^+(\theta) \hat{R}(\theta) = \hat{R}(-\theta) \hat{R}(\theta) = I. \quad (5.1)$$

Now one no longer has statistical models with real, non-negative Boltzmann weights. But the unitary matrices become relevant concerning entanglement. Such parametrizations of entangled states will be studied in a following paper.

The unitary $(2n)^2 \times (2n)^2$ braid matrices generate entangled quantum states with one difference as compared to the odd dimensional case. In the last section of Ref. [2] it was pointed out that the product of pure states $|0\rangle|0\rangle$ conserved its status under action of $(2n+1)^2 \times (2n+1)^2$ unitary matrix. For the present case there is no such exceptional state.

As already pointed out for odd dimensions (see section 5, Ref. [2]), even dimensional, unitary, multiparameter braid matrices are also periodic or quasiperiodic in θ accordingly as the m 's are mutually commensurate or not.

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